

# Spectrum of gravitational waves in Krein Space Quantization

M. Mohsenzadeh<sup>1\*</sup>, A. Sojasi<sup>2†</sup> and E. Yusofi<sup>3‡</sup>

February 23, 2012

<sup>1</sup> *Department of physics, Qom branch, Islamic Azad University, Qom, Iran*

<sup>2</sup> *Department of physics, Rasht branch, Islamic Azad University, Rasht, Iran*

<sup>3</sup> *Department of physics, Ayatollah Amoli branch, Islamic Azad University, Amol, Iran*

## Abstract

The main goal of this paper is to derive the primordial power spectrum for the scalar perturbations generated as a result of quantum fluctuations during an inflationary period by an alternative approach of field quantization [1, 2, 3]. Formulae are derived for the gravitational waves, special cases of which include power law inflation and inflation in the slow roll approximation, in Krein space quantization.

Keywords: power spectrum, Krein space quantization, gravitational waves.

*Proposed PACS numbers:* 04.62.+v, 03.70+k, 98.80.-k, 04.30.-w

## 1 Introduction

In this paper we derive formula for the spectrum of gravitational waves produced during inflation, for inflation in general in Krein space quantization. The standard results to first order in the slow roll approximation in Hilbert space quantization are

$$P_{\psi}^{1/2} = \frac{H}{2\pi}|_{aH=k} \quad (1)$$

for the gravitational wave spectrum[4]. Thus, the outline of the paper is as follows. In section 2 we briefly recall definition of power spectrum of gravitational waves. Section 3 is devoted to calculation of power spectrum in Krein space quantization. Section 4 is devoted to calculation of power spectrum in special cases. Brief conclusion and outlook are given in final section.

---

\*e-mail: mohsenzadeh@qom-iau.ac.ir

†e-mail: sojasi@iaurasht.ac.ir

‡e-mail: e.yusofi@iaumol.ac.ir

## 2 Definition of power spectrum of gravitational waves

Our units are such that  $c = \hbar = 8\pi G = 1$ .  $H$  is the Hubble parameter,  $\phi$  is the inflaton field and a dot denotes the derivative with respect to time  $t$ . The background metric is[5]

$$ds^2 = dt^2 - a^2(t)d\mathbf{x}^2 = a^2(\eta)(d\eta^2 - d\mathbf{x}^2) \quad (2)$$

Tensor linear perturbations to (2) can be expressed most generally as[6]

$$ds^2 = a^2(\eta)[d\eta^2 - (\delta_{ij} + 2h_{ij})dx^i dx^j] \quad (3)$$

The spectrum of gravitational waves is defined by[7, 8]

$$h_{ij} = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \sum_{\lambda=1}^2 \psi_{\mathbf{k},\lambda}(\eta) \xi_{ij}(\mathbf{k}, \lambda) e^{i\mathbf{k}\cdot\mathbf{x}} \quad (4)$$

$$\langle 0 | \psi_{\mathbf{k},\lambda} \psi_{\mathbf{l},\lambda}^* | 0 \rangle = \frac{2\pi^2}{k^3} P_\psi \delta^3(\mathbf{k} - \mathbf{l}) \quad (5)$$

where  $\xi_{ij}(\mathbf{k}, \lambda)$  is a polarization tensor satisfying

$$\xi_{ij} = \xi_{ji}, \xi_{ii} = 0, k_i \xi_{ij} = 0 \quad (6)$$

$$\xi_{ij}(\mathbf{k}, \lambda) \xi_{ij}^*(\mathbf{k}, \mu) = \delta_{\lambda\mu} \quad (7)$$

It is also useful to choose

$$\xi_{ij}(-\mathbf{k}, \lambda) = \xi_{ij}^*(\mathbf{k}, \lambda) \quad (8)$$

## 3 Calculation in Krein space quantization

In this section we calculate the power spectrum of gravitational waves in Krein space quantization. First, we briefly recall the Krein space quantization. In the previous paper [12], we present the free field operator in the Krein space quantization. The field operator in Krein space is built by joining two possible solutions of field equations, positive and negative norms

$$\phi(\eta, \mathbf{x}) = \phi_p(\eta, \mathbf{x}) + \phi_n(\eta, \mathbf{x}), \quad (9)$$

where

$$\phi_p(\eta, \mathbf{x}) = (2\pi)^{-3/2} \int d^3\mathbf{k} [c(\mathbf{k}) u_{k,p} e^{i\mathbf{k}\cdot\mathbf{x}} + c^\dagger(\mathbf{k}) u_{k,p}^* e^{-i\mathbf{k}\cdot\mathbf{x}}],$$

$$\phi_n(\eta, \mathbf{x}) = (2\pi)^{-3/2} \int d^3\mathbf{k} [b(\mathbf{k}) u_{k,n} e^{-i\mathbf{k}\cdot\mathbf{x}} + b^\dagger(\mathbf{k}) u_{k,n}^* e^{i\mathbf{k}\cdot\mathbf{x}}],$$

and  $c(\mathbf{k})$  and  $b(\mathbf{k})$  are two independent operators. Creation and annihilation operators are constrained to obey the following commutation rules

$$[c(\mathbf{k}), c(\mathbf{k}')] = 0, \quad [c^\dagger(\mathbf{k}), c^\dagger(\mathbf{k}')] = 0, \quad [c(\mathbf{k}), c^\dagger(\mathbf{k}')] = \delta(\mathbf{k} - \mathbf{k}'), \quad (10)$$

$$[b(\mathbf{k}), b(\mathbf{k}')] = 0, \quad [b^\dagger(\mathbf{k}), b^\dagger(\mathbf{k}')] = 0, \quad [b(\mathbf{k}), b^\dagger(\mathbf{k}')] = -\delta(\mathbf{k} - \mathbf{k}'), \quad (11)$$

$$[c(\mathbf{k}), b(\mathbf{k}')] = 0, \quad [c^\dagger(\mathbf{k}), b^\dagger(\mathbf{k}')] = 0, \quad [c(\mathbf{k}), b^\dagger(\mathbf{k}')] = 0, \quad [c^\dagger(\mathbf{k}), b(\mathbf{k}')] = 0. \quad (12)$$

The vacuum state  $|\Omega\rangle$  is then defined by

$$c^\dagger(\mathbf{k})|\Omega\rangle = |\mathbf{1}_\mathbf{k}\rangle; \quad c(\mathbf{k})|\Omega\rangle = 0, \quad (13)$$

$$b^\dagger(\mathbf{k})|\Omega\rangle = |\bar{\mathbf{1}}_\mathbf{k}\rangle; \quad b(\mathbf{k})|\Omega\rangle = 0, \quad (14)$$

$$b(\mathbf{k})|\mathbf{1}_\mathbf{k}\rangle = 0; \quad c(\mathbf{k})|\bar{\mathbf{1}}_\mathbf{k}\rangle = 0, \quad (15)$$

where  $|\mathbf{1}_\mathbf{k}\rangle$  is called a one particle state and  $|\bar{\mathbf{1}}_\mathbf{k}\rangle$  is called a one “unparticle state”. By imposing the physical interaction on the field operator, only the positive norm states are affected. The negative modes do not interact with the physical states or real physical world, thus they can not be affected by the physical interaction as well.

The action for tensor linear perturbation is[9]

$$S = \frac{1}{2} \int a^2 [(h'_{ij})^2 - (\partial_l h_{ij}^2)] d\eta d^3\mathbf{x} = \frac{1}{2} \int d^3k \times \sum_{\lambda=1}^2 \int [|v'_{\mathbf{k},\lambda}|^2 - (k^2 - \frac{a''}{a}) |v_{\mathbf{k},\lambda}|^2] d\eta \quad (16)$$

where

$$v_{\mathbf{k},\lambda} = a\psi_{\mathbf{k},\lambda} \quad (17)$$

[N.B.  $v_{\mathbf{k},\lambda} = v_{-\mathbf{k},\lambda}^*$  from (4) and (8)]. Quantizing

$$v_{\mathbf{k},\lambda}(\eta) = \{v_{k,p}(\eta)c(\mathbf{k},\lambda) + v_{k,p}^*(\eta)c^\dagger(-\mathbf{k},\lambda)\} + \{v_{k,n}(\eta)b(\mathbf{k},\lambda) + v_{k,n}^*(\eta)b^\dagger(-\mathbf{k},\lambda)\} \quad (18)$$

The equation of motion for  $v_k$  is

$$v_{\mathbf{k}}'' + (k^2 - \frac{a''}{a})v_{\mathbf{k}} = 0 \quad (19)$$

and

$$v_k(\eta) \sim \frac{1}{\sqrt{2k}} e^{\pm ik\eta}, \quad as \quad \frac{k}{aH} \gg 1 \quad (20)$$

$$v_k \sim a, \quad as \quad \frac{k}{aH} \ll 1 \quad (21)$$

Assuming  $\epsilon \equiv -\frac{\dot{H}}{H^2}$  is constant, and

$$\frac{a''}{a} = 2a^2 H^2 (1 - \frac{1}{2}\epsilon) = \frac{1}{\eta^2} (\mu^2 - \frac{1}{4}) \quad (22)$$

where  $\mu = \frac{1}{1-\epsilon} + \frac{1}{2}$ . Now

$$\langle 0 | \psi_{\mathbf{k},\lambda} \psi_{\mathbf{l},\sigma}^* | 0 \rangle = \frac{1}{a^2} (|v_{k,p}|^2 + |v_{k,n}|^2) \delta_{\lambda\sigma} \delta^3(\mathbf{k} - \mathbf{l}) \quad (23)$$

Therefore from (5) and (23)

$$P_\psi(k) = \frac{k^3}{2\pi^2} \frac{|v_{k,p}(\eta)|^2 + |v_{k,n}(\eta)|^2}{a^2} \quad (24)$$

## 4 Special cases

### 4.1 slow-roll approximation

The orthogonal eigenmodes  $u_k, u_k^*$  of (19) are easy to construct in the slow roll approximation, when  $\epsilon$  and  $\mu$  are small,  $\epsilon, \mu \ll 1$ . During slow roll inflation, to order in  $\epsilon$  and  $\mu$ , we have  $(1 - \epsilon)\eta = -\frac{1}{aH}$  and so  $\frac{a''}{a} = \frac{2-3\mu+6\epsilon}{\eta^2}$ . Then the mode equation (19) become[8, 10]

$$u_k'' + (k^2 - \frac{2-3\eta+6\epsilon}{\eta^2})u_k = 0 \quad (25)$$

The standard choice[11] of the eigenmodes  $u_k, u_k^*$  is to take

$$u_k(\eta) = -\frac{\sqrt{\pi\eta}}{2}H_\nu^{(-)}(k\eta), u_k^*(\eta) = -\frac{\sqrt{\pi\eta}}{2}H_\nu^{(+)}(k\eta) \quad (26)$$

as the positive and negative frequency modes, respectively, where  $\nu = \frac{3}{2} - \mu + 2\epsilon$ [8]. The normalization of  $u_k$  is chosen such that eqs.(10)-(15) follows from the canonical commutation relations  $[u(\eta, \mathbf{x}), \Pi(\eta, \mathbf{x}')] = i\delta^3(\mathbf{x} - \mathbf{x}')$ . Using  $|0\rangle$  as the state of the inflaton during inflation and ignoring the slow roll corrections, in which case the eigenmodes (26) reduce to

$$u_{k,p} = \frac{1}{\sqrt{2k}}(1 - \frac{i}{k\eta})e^{-ik\eta}, u_{k,n} = \frac{1}{\sqrt{2k}}(1 + \frac{i}{k\eta})e^{ik\eta} \quad (27)$$

Then, according to the second perspective of [12]

$$\langle 0 | \psi_{\mathbf{k},\lambda} \psi_{\mathbf{l},\lambda}^* | 0 \rangle = \frac{H}{2k^3} (ke^{-\alpha k^2}) \delta^3(\mathbf{k} - \mathbf{l}) \quad (28)$$

Therefore we have for power spectrum from (5) and (28)

$$P_\psi(k) = \frac{H}{4\pi^2} k e^{-\alpha k^2} \quad (29)$$

where  $\alpha = \frac{1}{\pi H^2}$ , is related to the density of gravitons[3, 13].

### 4.2 Power-law inflation

In power law inflation for the mode equation (19) we need  $a''/a$ . To compute this we have

$$a(t) \propto t^p \quad (30)$$

Then  $t \propto \eta^{1/1-p}$ , So  $a(\eta) \propto \eta^{p/1-p}$ . Hence,

$$\frac{a''}{a} = (\nu^2 - \frac{1}{4}) \frac{1}{\eta^2} \quad (31)$$

where

$$\nu^2 - \frac{1}{4} = \frac{p(2p-1)}{(p-1)^2} \quad (32)$$

Using this in (19) gives the mode equation

$$u_k'' + (k^2 - \frac{\nu^2 - 1/4}{\eta^2})u_k = 0 \quad (33)$$

This can be solved in terms of Bessel functions. Before proceeding with this we note two further relations. First, from  $H = \frac{p}{t}$  and  $a(t) = a_0 t^p$  we get

$$\eta = -\frac{1}{aH} \frac{1}{1 - 1/p} \quad (34)$$

In addition

$$\epsilon = \frac{1}{p} = \text{constant} \quad (35)$$

Let us now turn to the mode equation (33). According to [14], the functions  $\omega(z) = z^{1/2}c_\nu(\lambda z)$ ,  $c_\nu \propto H_\nu^{(1)}, H_\nu^{(2)}, \dots$  satisfy the differential equation

$$\omega'' + (\lambda - \frac{\nu^2 - 1/4}{z^2})\omega = 0 \quad (36)$$

From the asymptotic formula for large  $k\eta$  [14]

$$H_\nu(k\eta) \sim \sqrt{\frac{2}{\pi k\eta}} [1 - i \frac{4\nu^2 - 1}{8k\eta}] \exp[-ik\eta - i\pi(\frac{\nu}{2} + \frac{1}{4})] \quad (37)$$

According to the second perspective of [12], we see that the correct solutions are

$$v_{k,p}(\eta) = \frac{1}{\sqrt{2k}} (1 - \frac{i}{k\eta}) e^{i(\mu + \frac{1}{2})\frac{\pi}{2}} e^{-ik\eta} \quad (38)$$

$$v_{k,n}(\eta) = \frac{1}{\sqrt{2k}} (1 + \frac{i}{k\eta}) e^{-i(\mu + \frac{1}{2})\frac{\pi}{2}} e^{ik\eta} \quad (39)$$

where  $\mu$

$$\mu = \frac{3}{2} + \frac{1}{p-1} \quad (40)$$

Therefore from (38),(39) and using (5),(23) we have:

$$P_\psi(k) = \frac{H}{4\pi^2} k e^{-\alpha k^2} \quad (41)$$

## 5 Conclusion

The negative frequency solutions of the field equations are needed for the covariant quantization in the minimally coupled scalar fields in de Sitter space. Contrary to the Minkowski space, the elimination of de Sitter negative norms in this case breaks the de Sitter invariance. In other words, in order to restore the de Sitter invariance, one needs to take into account the negative norm states *i.e.* the Krein space quantization. This provides a natural tool for renormalization of the theory [1]. The spectrum of gravitational waves, has been calculated through the Krein space quantization exhibiting. Once again the theory is automatically renormalized.

**Acknowledgements:** This work has been supported by the Islamic Azad University-Qom Branch, Qom, Iran.

## References

- [1] J.P. Gazeau, J. Renaud, M.V. Takook, *Class. Quan. Grav.*, 17(2000)1415, gr-qc/9904023
- [2] M.V. Takook, *Int. J. Mod. Phys. E*, 11(2002)509, gr-qc/0006019
- [3] S. Rouhani and M.V.Takook, *INT. J. Theor. Phys.* 48 (2009)2740-2747.
- [4] D.H.Lyth and E.D.Stewart, *phys.Lett.B*274(1992)168.
- [5] A.R.Liddle and D.H.Lyth, "Cosmological inflation and large-scale structure", Cambridge University Press (2000).
- [6] J.M.Bardeen, *Phys.Rev.D* 22 (1980) 1882.
- [7] E.W.Kolb and M.S.Turner, *The early universe (Addison – Wesley, NewYork, 1990)*.
- [8] E.D.Stewart and D.H.Lyth, *phys.Lett.B*302(1993)171-175.
- [9] V.F.Mukhanov, H.A.Feldman and R.H.Brandenberger, *Phys.Rep.* 215 (1992) 203.
- [10] Nemanja Kaloper and Manoj Kaplinghat, *Phys. Rev. D* 68, 123522(2003), hep-th/0307016.
- [11] A. Linde, "Particle physics and inflationary cosmology", Harwood Academic,(1990).
- [12] M.Mosenzadeh, S.Rouhani, M.V.Takook, *INT. J. Theor. Phys.* 48 (2009)755.
- [13] Ford H.L., *Quantum Field Theory in Curved Spacetime*, gr-qc/9707062.
- [14] M.Abramowitz and I.Stegun, *Handbook of Mathematical Functions, with Formulas, Graphs, and Mathematical Tables*, Dover (1974).